On Tempo and Quantum

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This paper presents a general theory of the tempo and quantum of life events based on the notion of infinitesimally short life stages which successively cover the life span of every cohort. The tempo effect is shown to appear due to the different exposure of birth cohorts and the synthetic cohort to similar life stages. The distortions are neither linked to the type of events, cohort or period perspective nor to the type of rates describing the demographic process. The notion of postponement/anticipation of events is not a precondition for the tempo effect. We propose adjustment formulas for distortions at individual age groups, which correct such integral demographic indicators as the TFR when they are integrated over the life span. Our approach generalizes the results by Ryder, Bongaarts-Feeney and Kohler-Philipov without using their simplifying assumptions. The paper is supplemented by empirical example.

INTRODUCTION

Since the classical works by Ryder (1951, 1956, 1959, 1964, 1980), who advocated the cohort approach and proposed the translation equation to estimate cohort completed fertility from the period TFR, and by Bongaarts and Feeney (1998, 2002, 2003, 2006, 2008), who developed more realistic and an entirely period approach, the topic of distortions of period estimates of quantum due to changes of the tempo of life events has attracted wide attention in both methodological works (e.g. Kohler and Philipov 2001; Van Imhoff 2001; Yi and Land 2001, 2002; Kohler and Ortega 2002; Wachter 2005; Wilmoth 2005; Feeney 2006; Goldstein 2006; Guillot 2006; Rodriguez 2006) and applied ones (e.g. Sobotka 2004; Winkler-Dworak and Engelhardt 2004; Luy 2006). The phenomenon was shown to apply not only to fertility measures, but also to mortality and marriage measures. Bongaarts and Feeney (2006) advocated the relevance of the concept to any kind of life cycle events. Despite the considerable development in methodology and the experience gained in applications, however, the topic remains controversial. Questions related to whether it is justifiable and how to extend the concept beyond the traditional area of fertility, or what improvements could be made to the formulas by Ryder and Bongaarts-Feeney (which were derived from simplified assumptions) are still not resolved. Other relevant methodological questions include: what kind of rates to use, how to derive integral indicators from age-specific rates, how to interpret the adjusted quantum, what—period or cohort?—indices to build the theory on, etc. (see, e.g., Van Imhoff (2001).)

This paper presents a new approach to the problem based on a new geometrical interpretation. Our theory enables studying tempo and quantum *locally*, i.e., for small areas on the Lexis surface without assuming shifts or other simplified transformation of the whole age schedule of life cycle events. Neither have we assumed explicitly postponement or anticipation of demographic events from the remote past or future. We consider infinitesimal stages of life cycle of birth cohorts and present tempo changes as shifts of the life stages along the age scale. Exact definition and interpretation of life stages may vary depending on the research context. We show that the tempo-effect is caused by different exposure of birth cohorts and the synthetic cohort to shifting life stages, while their exposure is similar in the absence of the tempo change. In this framework, many methodological questions like those mentioned above may be separated from the phenomenon of tempo-effect as such.

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The paper consists of six sections. First, we provide discussion of the life stage, tempo, and quantum. We show that previous theories may be overarched within the single framework involving the notion of the life stage. Readers interested in formal results or in practical applications may skip most of the first section. In the second section, we present a general geometrical result, which demonstrates the essence of the tempo distortion caused by different exposure of the synthetic cohort and of birth cohorts to similar life stages. We proposed general formula for adjusting the age-specific rates. In the next section we develop further general relations and present tempo adjustments in terms of the stage function, which quantifies the concept of the life stage. Fourths section introduces the quantum function, which formalizes the notion of stage-specific subtotals of the quantum and is necessary for further derivations. Fifth section contains results important for practical applications. We consider two particular cases, when the quantum function is affected to cohort- or period-specific factors. In the first case, we develop general translation theory, while in the second case we formalize the Bongaarts-and-Feeney's concept of period conditions and propose generalizations to results by Bongaarts, Feeney, Kohler, Philipov, and Guillot. The last section contains illustrative application to adjusting the first order fertility rate in Czech Republic.

1. Basic definitions: Life stage, Tempo, and Quantum

We consider here the life course of birth cohorts to be split into a set of (indefinite number of) infinitesimally short successive *life stages*, which do not overlap with each other. In practice, the set of life stages may be defined in different ways, based on formal or substantive considerations, see some suggestions further down in the paper. (Main theoretical results derived here, however, remain valid irrespective to how exactly the life stages are defined.)

The definition of life stages must be linked to the demographic problem under study. One possibility to establish such link is through the definition of the *tempo*: we consider the tempo of events under study as being equivalent term for distribution of life stages along the age scale. Therefore, any change in positioning of life stages along the age scale is a *tempo change*. The very concept of postponement or advancement implies shifting along the age scale of certain life conditions of birth cohorts; the notion of life stages enables formalizing this concept. Unlike the theories proposed in the literature, we do not assume explicitly the postponement or advancement of demographic events as such. Traditional approach based on the concept of shifting events could be contradictive as it implies shifts of non-existing events when tempo changes are accompanied by changes of the quantum. Tempo theories known from the literature appeal to micro-level considerations of postponement and anticipation of events, despite being purely macro-level theories. Notion of life stages, which characterize the life cycle of cohorts, not of individuals, allows building the theory consistently on a macro-level.

We assume that the life stages are experienced only once by each of the birth cohorts. We also assume that the life stages may be observed only once at each of the calendar periods, i.e., two birth cohorts may not pass through the same stage at the same time. This limits the tempo change in such a way, that the natural sequence of life stages will not be interrupted due to advancement (postponement) of life stages. Together, these two assumptions imply that both the birth cohorts and the synthetic cohort experience the same sequence of life stages, albeit with a different exposures to similar life stages. (These assumptions may, in fact, be avoided by integrating actual exposures of birth cohorts and the synthetic cohort to different life stages in the Lexis surface; we do not consider such generalization here.)

Based on the life stage concept, we extend the notion of the *quantum*. Traditionally, the quantum is the number of events of interest over the life course of the birth cohort. Despite this strict cohort interpretation, the concept was implicitly applied in purely period theories. Interpreting the tempo-distortion of the period TFR, e.g., as a distortion of quantum, assumes period- not cohort-specific quantum. Such concept, however, may lack a strong logical justification, as the synthetic cohort does not represent experience of any birth cohort. Bongaarts and Feeney (1998, 2006) avoided direct reference to the quantum and proposed the notion of *period conditions* to be measured by the tempo-adjusted demographic indicators. This way, however, one may come to a relativistic conclusion, eroding the very foundation of the tempo theory, that "every measure measures something" (Wachter 2005). We avoid these complications by considering stage-specific subtotals of the quantum (stage-wise intensities of events in the continuous model). For any given elementary life stage, then, we show that the numbers of events obtained from period observation and from cohorts' experience will differ, whenever, the life stage shifts along the age scale. This difference is interpreted as tempo-distortion, which, thereby, has a strict traditional meaning: it indicates that period observations provide wrong conclusions about numbers of events experienced by birth cohorts. These distortions, however, are a purely period phenomenon, as we consider arbitrarily short elementary life stages and do not assume information about remote future or past. Although, our adjustments are, indeed, *translations* in Ryder's sense, we base the theory on *local* translations, which convert period numbers into cohort numbers of events within only infinitesimally short life stages observed at the moment. In particular, this enables us both to extend the Ryder's translation theory, and to formalize Bongaarts-and-Feeney's concept of *period conditions*.

Early works (e.g., Sauvy 1948; Hajnal 1947, 1950; Ryder 1951) considered postponement or anticipation of births due to external conditions, such as economic depression. They treated the completed cohort fertility as a stable indicator to be assessed by the means of *translation* of the period fertility (Ryder 1964). In such research context, the cumulated proportion of births given in a birth cohort by age x may be considered as indication of a certain life stage reached by the cohort; with positioning of the life stages along the age scale being adjusted according the varying external conditions. Such formally defined life stages are not necessarily linked to physiological, ideational or other substantive conditions; rather, they only indicate the completeness of cohort experience by age. The theory presented further down shows that such definition is natural for addressing the task of translating period rates into cohort totals and leads to the general translation formula.

The seminal work by Bongaarts and Feeney (1998) focused on period effects on fertility and enabled addressing tempo-effects without consideration of the whole life cycles of birth cohorts. The work starts with a stylized model, when "all women in every birth cohort have their (first) births at a single exact age". In our model, one may consider the age at which women have their births as belonging to a certain (infinitesimally short) stage in their life. Change in the age at birth is, then, interpreted as shift of the life stage along the age scale, and, as we show further down, the tempo distortion is a consequence of different exposures of birth cohorts and the synthetic cohort to the same life stage. Such interpretation is less controversial than the usual interpretation of change of age at childbearing as a reflection of postponement/anticipation of births as such: when tempo change is combined by change in the quantum, the usual interpretation would imply 'shifts' of non-existent births. A better interpretation would be that these are inherent age-specific conditions of life of birth cohorts, and not the births as such, which are subject to tempo change; while the number of births is a result of interaction of those age-specific cohort conditions with other determinants of the quantum, such as the period conditions. The concept of life stage enables formalizing these age-specific cohort conditions. Similar argumentation also applies to a more realistic situation considered by Bongaarts and Feeney, when births may occur at any age.

A more general model by Kohler and Philipov (2001) considers age-specific cumulated tempo $R(a,t)$. The model assumes "that births that occur at age a at time t would have occurred at an age $\alpha = a - R(a,t)$ at time $\tau = t - R(a,t)$ if there had been no postponement of fertility". Substituting the postponement of births by postponement of life conditions of birth cohorts, we reformulate the theory in terms of life stages being postponed by the cumulated tempo $R(a,t)$. Again, such interpretation is clearer and less contradictive in the usual case of changing quantum, as it does not imply shifting of non-existent events.

A limitation of the theory by Kohler and Philipov is due to consideration of postponement/anticipation between distant ages, which explicitly links the current conditions to the remote past or future. When population experiences profound change in life conditions, it might be implausible to consider events after the change as being postponed from the period prior to the change. Explicit consideration of remote past and future is limitation to a theory of period fertility. The theory presented here refers to life stages characteristic only for the period of observation and, thereby, avoids considering explicitly effects of the distant past and future on current estimates (similar to the theory by Bongaarts and Feeney).

Considerations similar to those presented for the model by Kohler and Philipov also apply to the model by Kohler and Ortega (2002) and to other models involving cumulated shifts of life cycle events (Yi and Land 2002; Feeney 2006; Goldstein 2006; Rodriguez 2006).

Wilmoth (2005), although arguing against adjusting mortality rates for the tempo effect, proposed age-specific adjustments generalizing the Bongaarts-Feeney theory of mortality tempo. His results are identical to those implied by our theory under particular assumption that life cycle events are only subject to tempo change, as it is usually accepted for the case of mortality.

Hence, the notion of life stages enables to reformulate and overarch in a single theory the previous results. In addition, we generalize Ryder's translation theory and show that some results by Bongaarts and Feeney, by Kohler and Philipov, and by Guillot are valid and may be extended under wider conditions than those originally assumed.

In general, the notion of life stages enables more transparent argumentation about the tempo effects. In case of mortality, for example, an important aspect of the phenomenon might have been overlooked in the literature, while it becomes evident within the framework of life stages theory. Namely, the usual way of adjustment implies possibly artificial compression of durations of exposure of people to the prevailing mortality levels, which explains somewhat counter-intuitive downward jump of life expectancy under 'no tempo' scenario (I intend to address this issue in more details elsewhere).

The concept of life stages is also useful in its ability to incorporate substantive knowledge about age-specific conditions of birth cohorts. Age at completing the education, for example, was argued to have a strong impact on timing of births (Blossfeld and Huinink 1991; Rindfuss *et al.* 1996). Such information may be taken into account by linking life stages at the early childbearing period to the age at completing the education. Although, the substantive knowledge may not provide the whole structure of the life stages, our theory provides ways to combine fragmentary substantive knowledge about developments of particular turning points in the life cycle with formal consideration of the life stages in between those turning points.

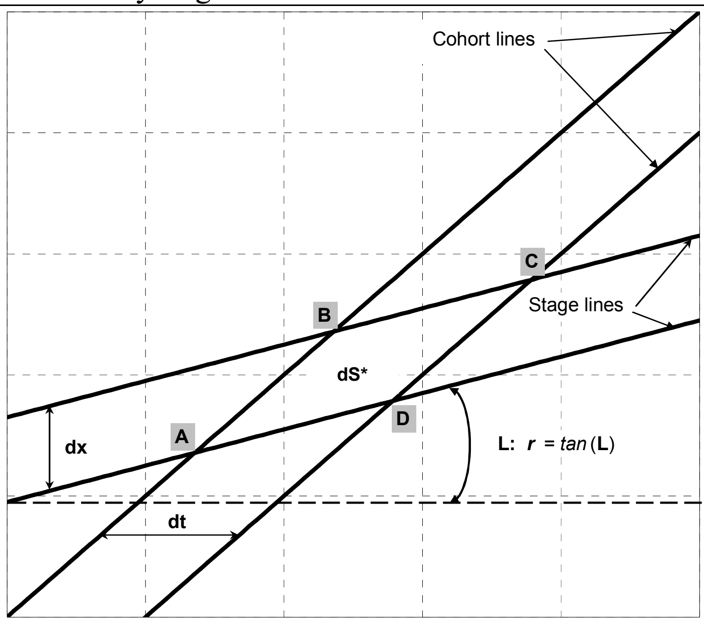
Although the life stages may not be directly observed, we arrive at practical methods of reconstructing the structure of life stages based entirely on the dynamics of demographic rates.

2. GENERAL THEORY AND ADJUSTMENT FORMULA: A LOCAL PERSPECTIVE

As described above, we consider the life course to be split into a set of successive *life stages*. We assume that it is possible to define the structure of life stages so precisely that the (indefinite number of) infinitesimally short life stages or, equivalently, ages marking the transition from one stage to another fill the entire age axis. We also assume that a life stage may be experienced only once during the life course of a birth cohort and may be observed only once at a calendar time.

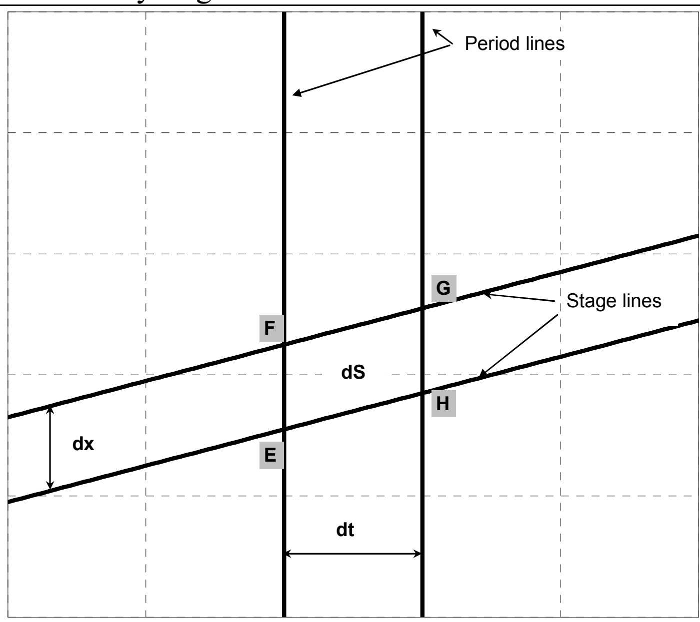
Such representation of the life course may be illustrated in the Lexis diagram. Consider the Lexis diagram, where the horizontal axis represents calendar time and the vertical axis represents age; hence, birth cohorts are represented by diagonal lines originating at time of birth of the cohort. In addition to the traditional lines representing life course in the age-period space, we introduce the *stage lines*; each stage line showing by what age at each calendar time a birth cohort reaches a certain life stage. Hence, for each birth cohort, age/time at crossing a particular stage line indicates the moment when the cohort enters into the corresponding life stage. The stage lines may not intercept, as a birth cohort may experience only one life stage at a moment.

Figure 1a Elementary area of the Lexis surface bounded by stage and time lines



$$S_{ABCD} = dS^* = \frac{dxdt}{1-r(x;t)}$$

Figure 1b Elementary area of the Lexis surface bounded by stage and cohort lines



$$S_{EFGH} = dS = dxdt = dS^* \cdot \left(1-r(x;t)\right)$$

Two such stage lines along with two cohort lines are shown in Figure 1a. The stage lines shown in the figure bound an elementary life stage of length *dx* along the age scale. The lower stage line in the figure indicates when birth cohorts enter the elementary life stage and the upper line indicates when cohorts leave the elementary life stage to the next stage (not shown in the figure). The earlier born exact birth cohort shown in the figure enters the elementary life stage on reaching the point A and leaves it to the next stage on reaching the point B. The later born cohort, correspondingly, enters in and leaves out of the elementary life stage at points D and C. Since the stage lines shown in the figure have a positive slope, every successive cohort stays at the elementary life stage at more advanced age compared to the preceding cohorts; i.e., the picture indicates existence of postponement. Would there be anticipation experienced by the population, the stage lines would have negative slope. Horizontal stage lines would indicate absence of tempo changes, as every successive cohort would pass through the life stage at the same age as the preceding cohort.

The two exact cohort lines shown in the figure bound an elementary birth cohort born during time interval *dt*. The life cycle events, which happen to this elementary cohort while it is passing through the elementary life stage indicated in the figure, are represented in the Lexis surface by the quadrangle ABCD; the area of this quadrangle determines exposure of the cohort to the elementary life stage.

Figure 1b contains the same stage lines, which are accompanied by time—not cohort—lines bounding an elementary observation period of the same duration *dt* as the one determining the elementary birth cohort above.

From the cohort perspective, the number of events within the life stage experienced by the elementary birth cohort is proportional to the elementary exposure area ABCD in figure 1a. This area equals $dS^* = \frac{dxdt}{1-r}$, where r is the tangent slope of the stage lines bounding the elementary area (see Appendix 1). By contrast, being observed within the calendar period, the number of events within the same life stage will be proportional to another area EFGH (figure 1b), which equals $dS = dxdt$ (see Appendix 1). If the stage lines have positive or negative slope, $r \neq 0$, the two areas will be different, and their ratio $1-r$ gives the distortion caused by different exposure of the birth cohort and the synthetic cohort to the elementary life stage. Our assumptions about the life stages' occurrences within cohorts and periods guarantee that the geometrical considerations above are sound: the slope of stage lines may not exceed unity; neither may the stage lines reverse their direction in the Lexis surface.

Geometrical considerations above lead to a simple and, yet, fundamental observation: *whenever there is a shift of age at which people enter a certain life stage, exposures to the life stage observed from the hypothetical cohort are not consistent with exposures of real birth cohorts to the same life stage.*

One consequence of this exposures' mismatch is that the number of events of any kind that is observed within the life stage[2] is different depending on whether it was obtained from the history of a real birth cohort or from period observation over adjacent birth cohorts. (This applies both to the actual numbers of events and to the numbers standardized by the cohort size, by the population size, by the population at risk, etc.) Distorted estimates of the numbers of events at different life stages may distort both the estimates of the total amount of events over the life span (e.g., the total fertility rates) and the characteristics of distribution of events along the age scale (e.g., the indicators of life expectancy). Rates and indicators of any kind will be subject to distortions, as the tempo effect distorts the very original counts of events.

The distortion between period and cohort estimates of the numbers of life cycle events within infinitesimally short life stages may be corrected for by the following adjustment:

$$e^*(x;t) = \frac{e(x;t)}{1-r(x;t)}, \tag{1}$$

where $e(x;t)$, $e^*(x;t)$ are the observed and adjusted densities of events and $r(x;t)$ is the slope of the stage line at point $(x;t)$ in the Lexis surface. For an elementary life stage, adjustment based on (1) is *translation* of the period number of events into the cohort number

[2] To facilitate the reading, we refer to numbers of events within elementary life stages. In the continuous theory, a more formal approach is based on the quantum function introduced in section 4. However, one may also consider numbers of events within life stages short enough to assume uniform distribution of events within the stage.

of events during the life stage. Being integrated over a significantly long subset of the life span Ω

$$E^*_{\Omega} = \int_{\Omega} e^*(x;t)dx = \int_{\Omega} \frac{e(x;t)}{1-r(x;t)}dx \qquad (2)$$

or over the entire life span, however, the adjusted intensity provides the true synthetic summary of numbers of events at life stages of birth cohorts, which fall into the observation period. Under different conditions, the adjusted subtotal (2) may: represent translation into the cohort quantum; indicate the period total of events to be observed under the no tempo changes scenario; or reflect how favorable the period conditions are for the intensity of events at life stages which happen to fall within the period (see Ediev 2008 and further down in this paper).

When the shift of life stages is the same at all ages, the denominator in (2) is age-independent and we arrive at the distortion of the number of events over the entire life span, which was considered by Bongaarts and Feeney. Assuming, additionally, changing spread of life stages, we arrive at adjustments similar to those proposed by Kohler and Philipov. Our geometrical interpretation, however, shows that the tempo-effect is a phenomenon of local nature and integrates into classical distortions driven by changes in the overall distributional characteristics only in special cases. The age-specific character of the phenomenon was already pointed out by Kohler and Philipov (2001), who used the cumulated amount of postponement of births. Similar concepts were also applied by Wilmoth (2005) and Feeney (2006) to mortality tempo. In case when events under study are only subject to tempo change, without change in the quantum, these theories are equivalent to the one presented here. Our theory implies, however, that any reference to such an integral postponement characteristics is unnecessary.

A note is due here on applying the adjustment above to processes, which are studied in terms of exposures and not in terms of the numbers of events. (Usually, such processes are studied by the means of the rates of the first kind (events to exposure).) Mortality represents an important example of such process. It is widely accepted (and used as an argument against tempo-adjustments in mortality) that age-specific mortality rates as such describe mortality conditions; and that the numbers of deaths (standardized by the cohort radix or not) are only products of the mortality rates by the population exposed. However, it follows from the results above that when age-specific rates change with time, the synthetic cohort and birth cohorts are exposed differently to similar rates. (To appreciate this, consider life stages linked to the levels of mortality.) Therefore, the true synthetic summary of the prevailing mortality conditions should include both the prevailing mortality rates and durations of exposure of real people to those rates. In order for the synthetic cohort to be a cross-section of true experiences of birth cohorts, the observed period exposures to different levels of mortality should be adjusted by a factor of $1/(1-r(x;t))$, with $r(x;t)$ being the tangent slope of the line in the Lexis surface, along which the mortality rate is constant. Such approach may reconcile both the pro- and anti- mortality-tempo arguments, as it appreciates, firstly, the observed mortality rates as describing the mortality conditions and, secondly, the existence of the tempo-distortion of the deaths' number due to mismatch between exposures of the synthetic cohort and of birth cohorts to similar mortality levels. Unlike the traditional adjustments, however, we do not adjust the mortality rates in order to fit the cohort number of deaths into the period exposures. Rather, we propose adjusting the exposures themselves, so that the observed rates imposed over adjusted exposure periods produce numbers of deaths consistent with cohorts' experience. This approach provides interesting insights into the mortality dynamics, reveals existence of the *momentum of mortality change*, and leads to new projection methods. I intend to address these aspects elsewhere. Here, the focus is on the general foundations of the tempo-theory.

3. Quantification of the life stages: the stage function. Further general adjustment formulas

We may quantify the life stages by assigning number *s* to each of the stage lines. In principle, the choice of *s*-values is rather arbitrary. It is convenient, however, to assume that these values increase monotonically from the value 0 at birth up to value 1 as the cohort (period schedule) eventually reaches the end of the life course[3]. (Once this property holds for cohorts, it must also hold for periods and vice versa, due to the assumptions above about the slope of the stage lines.) This assumption still leaves a wide range of options for the *s*-values: any monotonic transformation of them, with end values fixed at levels 0 and 1, results in alternative acceptable set of *s*-values. As stage lines do not intercept, numbers *s* assigned to them form functions of age and of birth cohort/of calendar time; we denote these cohort- and period-wise *stage functions* by $s(x,t)$ and $s(x;t)$ correspondingly. We separate the calendar time variable by a semicolon from the age variable, while for the cohort's date of birth we use the comma as a separator, i.e., by definition, $s(x,t) \equiv s(x;t+x)$ is the value assigned to the stage line crossed by cohort *t* at age *x*. We assume these functions to be smooth enough, i.e. to have all necessary derivatives.

Using the stage function it is possible to derive from (1) the following general adjustment formula (see Appendix 2):

$$e^*(x;t) = e(x;t)\frac{s'_x(x;t)}{s'_x(x,t-x)} = e(x;t)\frac{s'_x(x;t)}{s'_x(x;t)+s'_t(x;t)}. \qquad (3)$$

Note, denominator and enumerator assume taking derivative along the period and cohort respectively. Interpretation to (3) consistent with our geometrical interpretation above is the following: values $\frac{1}{s'_x(x,t-x)}$ and $\frac{1}{s'_x(x;t)}$ give relative exposures of the birth cohort and the synthetic cohort to the life stage attained at age *x* at time *t*; so that their ratio provides distortion factor of the period estimate of exposure to the life stage. The latter expression in (3) indicates that whenever there is a tempo change, $s'_t(x;t) \neq 0$, there is a distortion.

In case of fertility, Eq. (3) implies for the adjusted TFR:

$$TFR^*(t) = \int_0^\infty f^*(x;t)dx = \int_0^\infty f(x;t)\frac{s'_x(x;t)}{s'_x(x,t-x)}dx, \qquad (4)$$

where $f(x;t)$ is the observed age-specific fertility rate at age *x* at time *t*.

For practical purposes, relation (3) may be approximated by the finite differences:

$$e^*(x;t) \approx e(x;t)\frac{s\left(x+\frac{\Delta}{2};t\right)-s\left(x-\frac{\Delta}{2};t\right)}{s\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right)-s\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}, \qquad (5)$$

here Δ is the interval, over which the average derivative is approximated[4]. In fertility tempo studies, for instance, erratic variations may usually be prevented by setting $\Delta = 3$ [5]:

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[3] This assumption is only necessary for the following results for integrals of rates over the life span. The only precondition for the age-specific adjustments (3) is the existence of non-zero derivative $s_x'(x,t-x)$. More general adjustments may be based on integrals over finite subsets of cohort life spans, which we do not consider here.

[4] To facilitate practical usage, we present stage functions in (5) as functions of calendar time only using the relation: $s\left(x \pm \frac{\Delta}{2}, t-x\right) = s\left(x \pm \frac{\Delta}{2}; t \pm \frac{\Delta}{2}\right)$, which is evident from our comma-semicolon system of notations.

$$TFR^*(t) = \sum_x F_{xt}{}^* \approx \sum_x F_{xt} \frac{s\left(x+2; t+\frac{1}{2}\right) - s\left(x-1; t+\frac{1}{2}\right)}{s(x+2; t+2) - s(x-1; t-1)}, \tag{6}$$

where F_{xt} is the observed fertility rate at age x in year t [6].

Practical solutions for the stage function in the formulas above are provided in section 5. We show, in particular, that cumulated cohort- or period-wise proportions of life cycle events by different ages may be used as stage functions in some research contexts.

The formulas above are *general*; we imposed no specific assumptions regarding the dynamics of the demographic rates, except for rather non-limiting assumption that each life stage is observed only once in the synthetic cohort and in birth cohorts. Elsewhere, this profile was assumed either to shift (Ryder, Bongaarts and Feeney), to shift and change its variance (Kohler and Philipov; Kohler and Ortega) or to follow a specific model (Ryder; Yi and Land).

4. The quantum function

Common definition of the quantum in demography refers to the total number of events of interest during the life course of birth cohorts. We formalize the notion of the quantum on local (i.e., on stage- and age- specific) basis by introducing the quantum function as the number of life cycle events per unit change of the stage function; in the continuous theory, this can be done using derivative of the cumulated number of events over the stage function:

$$q(s,t) \stackrel{def}{=} \frac{d}{ds} \int_0^{x(s,t)} e(y,t)dy = e(x(s,t),t) \cdot x'_s(s,t) = \frac{e(x(s,t),t)}{s'_x(x(s,t),t)}, \tag{7}$$

here, $q(s,t)$ is the quantum function and $x(s,t)$ is the inverse function to the stage function $s(x,t)$ (the inverse function exists due to the assumed monotonicity of the stage function).

It is useful to express the quantum function in terms of age and cohort:

$$q(x,t) \stackrel{def}{=} q(s(x,t),t) = \frac{e(x,t)}{s'_x(x,t)} = \frac{e^*(x,t)}{s'_x(x;t+x)}, \tag{8}$$

or in terms of age and period:

$$q(x;t) \stackrel{def}{=} q(x,t-x) = \frac{e(x,t-x)}{s'_x(x,t-x)} = \frac{e^*(x;t)}{s'_x(x;t)}. \tag{9}$$

The cohort total of life cycle events, i.e., the traditional quantum, may, naturally, be obtained by integrating the quantum function over the life span of the cohort:

$$CE(t) = \int_0^\infty e(x,t)dx = \int_0^\infty e(x(s,t),t)x'_s(s,t)ds = \int_0^1 q(s,t)ds \tag{10}$$

(note usage of (7)), while the adjusted period total may be obtained by integrating the quantum function across the life stages observed in the period:

$$TE^*(t) = \int_0^\infty e^*(x;t)dx = \int_0^\infty q(x;t)s'_x(x;t)dx = \int_0^1 q(s;t)ds. \tag{11}$$

These formalizations are useful in what follows below when extending the theories by Ryder, Bongaarts and Feeney, Kohler and Philipov, and Guillot.

[5] Bongaarts and Feeney (2000) recommended a similar triennial approximation in their formula.

[6] We consider single-year age groups and calendar periods and approximate the stage functions by their values at the mid-year point $t+\frac{1}{2}$ for the year t.

5. Important particular cases: cohort-wise proportional and period-wise proportional quantum

Cohort-wise proportional changes in the quantum: the generalized Ryderian case. General translation formula

The early studies of the tempo-effect assumed existence of rather stable cohort totals of life cycle events, age distribution of which is reshaped due to varying period conditions. We may consider such situation by assuming a cohort-specific factor of the quantum function:

$$q(s,t) = CE(t) \cdot \alpha(s), \tag{12}$$

where $\int_0^1 \alpha(s)ds = 1$ by the scaling agreement and, then, $CE(t)$ are cohort totals of life cycle events as it follows from (10). Assumption (12) does not limit anyhow the tempo change, as it does not include the age variable explicitly and allows for arbitrary redistributions of life cycle events within the life cycle of birth cohorts. Such assumption is more general than the traditional assumption that tempo changes are induced only by period factors.

Assumption (12) immediately implies that the adjusted period total of life cycle events is a weighted average of the birth cohorts' totals:

$$TE^*(t) \overset{def}{=} \int_0^\infty e^*(x;t)dx = \int_0^\infty CE(t-x)w(x;t)dx, \tag{13}$$

where non-negative weights $w(x;t) = \alpha(s(x;t))s'_x(x;t)$ sum up to unity. This relation is consistent with the Ryder's interpretation of the adjusted period total as an approximation of the quantum averaged over the cohorts observed in the period.

Another general and practically useful result addresses the problem of observability of life stages: *the quantum function may be subject to cohort-specific proportional changes (12) if and only if the stage function may be defined as cumulated proportion of the events observed in the birth cohort.* To prove this, note that the cumulated proportion $p(x,t)$ of events by age x of cohort born at time t is a monotonic transformation of the stage function:

$$p(x,t) = \frac{\int_0^x e(y,t)dy}{CE(t)} = \frac{\int_0^{s(x,t)} q(u,t)du}{CE(t)} = \frac{CE(t)\int_0^{s(x,t)} \alpha(u)du}{CE(t)} = \int_0^{s(x,t)} \alpha(u)du = g(s). \tag{14}$$

Note the non-negativity of the slope of the transformation: $g'(s) = \alpha(s) \geq 0$; it may equal zero only for age groups in which no life cycle events occur. Therefore, cohort proportions $p(x,t) = g(s(x,t))$ may be used to produce the same adjustments as implied by $s(x,t)$, except for stages with zero quantum, which have no effect on the adjusted numbers of events.

This result implies that the following is the most general adjustment formula for the case of cohort-driven proportional changes in the quantum:

$$\begin{aligned} e^*(x;t) &= e(x;t)\frac{p'_x(x;t)}{p'_x(x,t-x)} = e(x;t)\frac{p'_x(x;t)}{p'_x(x;t) + p'_t(x;t)} \\ &\approx e(x;t)\frac{p\left(x+\frac{\Delta}{2};t\right) - p\left(x-\frac{\Delta}{2};t\right)}{p\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right) - p\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}. \end{aligned} \tag{15}$$

Such adjustment, in turn, provides the most general translation equation in the sense of Eq. (13):

$$\int_0^\infty CE(t-x)w(x;t)dx = \int_0^\infty e(x;t)\frac{p'_x(x;t)}{p'_x(x,t-x)}dx$$

$$\approx \sum_x e(x;t)\frac{p\left(x+\frac{\Delta}{2};t\right)-p\left(x-\frac{\Delta}{2};t\right)}{p\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right)-p\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}. \tag{16}$$

In some cases cohort-wise proportional changes in the quantum may not be applied to the whole life span. The quantum of fertility at young childbearing ages, in the middle of the childbearing or at 'late-fertility' ages may respond differently to a changing environment. A general case might be of practical importance, where the quantum changes proportionally within a part $s \in [s1;s2]$ of birth cohorts' life course only. Importantly, the relations presented above may be extended to the case of considering only parts of the life span based on cumulated cohort proportions of events within those parts of the life span (Ediev 2008).

Period-wise proportional changes in the quantum: the generalized Bongaarts-Feeney case. Formalization of the concept of period conditions

Instead of looking for a translation into cohort totals, Bongaarts and Feeney addressed favorability of the *period conditions* for the quantum of life cycle events. We formalize this approach by considering period-wise proportional changes of the quantum function[7]:

$$q(s;t)=Q(t)\cdot\alpha(s), \tag{17}$$

where $\int_0^1 \alpha(s)ds=1$ by the scaling agreement and, therefore, (11) implies that $Q(t)$ is the adjusted period total of life cycle events. Hence, the adjusted period total of life cycle events provides an estimate of the period-specific factor of the quantum function; in this sense, it is an indicator of favorability of *period conditions* for the quantum of life cycle events.

In the period-proportionality case (17)—opposite to the cohort-proportionality case considered above—the cohort quantum is a weighted average of the adjusted period totals:

$$CE(t)=\int_0^\infty Q(t+x)w(x,t)dx, \tag{18}$$

where non-negative weights $w(x,t)=\alpha(s(x,t))s'_x(x,t)$ sum up to unity.[8] Bongaarts and Feeney (2006) showed similar relation for the case where tempo distortions do not depend on age and the constant shape assumption holds true.

Importantly, *the quantum function may be subject to period-specific proportional changes (17) if and only if the life stages may be defined as the <u>adjusted</u> cumulated proportion of life cycle events in the calendar period.* (Proof is similar to that presented above with respect to the case of cohort-specific quantum factor; see Ediev 2008.)

Hence, the following is the most general adjustment formula for the case of period-wise proportional changes in the quantum function:

$$e^*(x;t)=e(x;t)\frac{p_x^{*\prime}(x;t)}{p_x^{*\prime}(x,t-x)}=e(x;t)\frac{p_x^{*\prime}(x;t)}{p_x^{*\prime}(x;t)+p_t^{*\prime}(x;t)}$$

$$\approx e(x;t)\frac{p^*\left(x+\frac{\Delta}{2};t\right)-p^*\left(x-\frac{\Delta}{2};t\right)}{p^*\left(x+\frac{\Delta}{2};t+\frac{\Delta}{2}\right)-p^*\left(x-\frac{\Delta}{2};t-\frac{\Delta}{2}\right)}, \tag{19}$$

here $p^*(x;t)$ is the adjusted proportion of events observed below age *x* at time *t*.

Assumption (17) leads to the following generalization to the Bongaarts-Feeney formula (Appendix 3; Ediev 2008):

[7] Similar assumption was already employed by Kohler and Philipov 2001, Eq. (2).
[8] Yet, this relation may have only a limited practical value due to high sensitivity of weights to data errors.

$$TE^*(t) = \frac{TE(t)}{1 - \frac{d}{dt}\mu^*(t)}, \tag{20}$$

where $\mu^*(t) = \int_0^X x \cdot \frac{e^*(x;t)}{TE^*(t)} dx$ is the mean age at life cycle event according the adjusted period distribution. Under the shifting assumption, the observed and adjusted mean ages are identical, and (20) turns into the Bongaarts-Feeney formula. A result similar to (20) was shown by Kohler and Philipov (2001) after imposing a specific assumption (Eq. (10) in the cited work) in addition to assumption similar to (17). Here we have shown that (20) is a direct consequence of assuming period-wise proportional changes of the quantum function.

Our next results show that there is an indefinite number of adjustments consistent with the period-proportionality of the quantum and the choice among them requires exactly one additional assumption. In order to establish this, note that for an *arbitrarily* assumed dynamics of the adjusted period total $TE^*(t)$ there is a corresponding age-specific adjustment (19) consistent with the period-proportionality and determined by the following stage function[9]:

$$p^*(x,t) = \int_0^x \frac{e(y;t+y)}{TE^*(t+y)} dy . \tag{21}$$

(This follows from (19) and from the obvious relation: $p_x^{*\prime}(x;t) = \frac{e^*(x;t)}{TE^*(t)}$.)

Formally, (21) implies an additional constraint, since the stage function should monotonically increase up to the value of 1 as age reaches the upper limit of the life span:

$$\int_0^\infty \frac{e(y;t+y)}{TE^*(t+y)} dy \equiv 1 . \tag{22}$$

However, (22) is equivalent to an *incorrect* Volterra's integral equation of the first kind (Ediev 2008) and requires *a priori* information for the solution. From the practical point of view, although an arbitrary choice for the adjusted period total in (21) may violate the constraints $s(x,t) \le 1$, $s(\infty,t) = 1$, such violations will take place at advanced ages, where intensities of events are low while the impact of data errors and of deviations from the proportionality assumption are high. Therefore, such violations: (i) are not significant; (ii) may be offset by corrections to the stage function $s(x,t)$, which may distort the actual dynamics of the adjusted period total $TE^*(t)$ from the one assumed by only a minor margin. Practical calculations show that Eqs. (21), (22)—even for the 'trivial' case of time-invariant schedule $e(x;t) \equiv e(x;t_0)$[10]—allow for nearly arbitrary choice of the desired level of the adjusted period total. In the case of fertility, e.g., it is possible to vary voluntarily the level of the adjusted TFR from less than 1% of the observed TFR to nearly as high as 100 times the observed TFR.

Hence, *the assumption of period-proportionality of the quantum function—unlike the cohort-proportionality assumption—is not sufficient to define the adjusted total and the age profile of life cycle events*. It allows for freedom of choice of the adjusted period total number

[9] In practical calculations, one should also apply corrections at advanced ages in order to assure monotonicity of the stage function and to prevent adjusted rates from turning negative.

[10] In this case one may solve Eq. (22) at $TE^*(t) \equiv TE(t)$. Yet, this 'no tempo change' solution is only one of many possibilities: alternatively, the tempo and quantum may change, while the rates are constant at dynamic equilibrium.

of life cycle events[11]. As the value of the adjusted period total $TE^*(t)$ may be chosen arbitrarily and is sufficient to define all the age-specific adjustments, one would need exactly one additional assumption or constraint in order to define the adjustment.

Unlike Eq. (22), Eq. (21) is of practical importance, as it allows deriving the age-specific adjustments from the assumed level of the adjusted period total of events:

$$e^*(x;t) = s'_x(x;t) \cdot TE^*(t) = TE^*(t) \cdot \frac{d}{dx}\left[\int_0^x \frac{e(y;t-x+y)}{TE^*(t-x+y)}dy\right]. \tag{23}$$

The Bongaarts-Feeney adjustment presents an important case where the extra assumption needed to define the adjustment is *the equality of change rates of the adjusted and the observed mean ages at life cycle events*. The traditional shifting assumption represents only one case of such type; validity of the Bongaarts-Feeney method does not rely on the shifting or any other assumptions concerning the tempo changes except for those related to the observed and adjusted mean ages. Kohler and Philipov (2001) proposed an improvement to the method by adding the effect of variance. Yet, as it follows from our results, the Bongaarts-Feeney method is consistent with the period-proportionality (17) also in the case of changing variance. The solution for this paradox lies in the fact that the Bongaarts-Feeney formula, when applied to data with arbitrary dynamics of the variance and supplemented by Eq. (23) to derive age-specific adjustments, will null the effects of the *adjusted* variance.

Kohler and Philipov (2001) provided another adjustment consistent with period-proportionality assumption. The necessary additional assumption is expressed in their method in terms of the cumulated tempo (Eq. (10) in their paper). That assumption is rather unrealistic as it implies, for instance, that the upper age limit for fertility increases faster than the mean age at childbearing. However, just like in the case of the Bongaarts-Feeney method, the results by Kohler and Philipov can be extended without using their original assumption. The following general equation is a direct consequence of the period-proportionality assumption (17):

$$\mu(t) = \mu^*(t) - \frac{\frac{1}{2}\frac{d}{dt}\sigma^{*2}(t)}{1-\frac{d}{dt}\mu^*(t)}, \tag{24}$$

where $\sigma^{*2}(t) = \int_0^X (x-\mu^*(t))^2 \cdot s'_x(x;t)dx = \int_0^X x^2 \cdot s'_x(x;t)dx - \mu^{*2}(t)$ is the variance of the adjusted period age distribution of events. The results established by Kohler and Philipov (2001, Result 8, Eq. (3) for linear changes and Result 10 for non-linear changes) are particular cases of (24) where the dynamics of the mean age and the variance is of special type (Eq. (10), Result 6 from the aforementioned paper). Using our theory one may also develop expressions for the effects of moments of higher order (Ediev 2008).

As in the cohort-wise proportionality case, the period-wise proportionality (17) may occasionally be considered too simplistic when applied to the whole life span. Young-age fertility, for example, may be related to the efficiency of contraception and to pre-marital sexual behavior, rather than to regulation of the timing of births. If so, births at young ages may be assumed to change their quantum rather than tempo in response to varying period conditions. At the same time, schedule of births in the middle of the fertile ages is more likely to be adjusted to the changing environment. Under such conditions, it might be more plausible to apply the period-wise proportionality (17) independently at separate parts of the life span. It is of practical importance, therefore, that the relations presented above—

[11] There will be no freedom of choice if life stages are defined from independent considerations, e.g. from cumulated (unadjusted) period proportions of events; this, however, may violate assumption (17).

including the formula by Bongaarts and Feeney—may be applied independently to separate parts of the life span (Ediev 2008) based on separately calculated proportions of events, mean ages at event, etc.

Time-invariant quantum function: life stages with fixed numbers of events

When the quantum function depends only on stage and not on cohort or period,

$$q(s,t) \equiv q(s), \tag{25}$$

one may apply all the results presented above for the cohort- and period-wise proportionality cases. Eq. (25) implies pure tempo changes with fixed distribution of the quantum between the life stages. Only in this particular case the dynamics of rates may be interpreted as a reflection of postponement/anticipation of life cycle events as it was widely considered in the literature (works on mortality tempo and elements of models of fertility tempo). Under this assumption, one may extend the following bridges from our results to those presented in the literature.

As the quantum function (25) does not depend on time, the cohort quantum and the adjusted period total are constant and equal. In works on mortality tempo this is reflected in the usual assumption that the adjusted Total Mortality Rate (TMR) must equal unity.

As (25) falls under the cohort- and period-wise proportionality simultaneously, the adjusted period proportion of events observed below age x equals the cumulated cohort proportion of events by the same age: $p^*(x;t) = p^c(x,t-x) = \int_0^{s(x,t)} q(u)du$. Therefore, the adjusted period mean age at life cycle event may be obtained by integrating over the cross-section of remaining cohort proportions of events:

$$\mu^*(t) = \int_0^\infty x \cdot \frac{e^*(x;t)}{TE^*(t)} dx = \int_0^\infty \left(1 - p^*(x;t)\right) dx = \int_0^\infty \left(1 - p^c(x,t-x)\right) dx \,. \tag{26}$$

(Transformation between the first two integrals is similar to the usual life table technique, when life expectancy may be presented either as mean age at death or as integral of the survival function.) In mortality studies, expressions in the last two integrals are the adjusted period and the cohort survival functions; the latter integral is what is usually called CAL (cross-sectional average length of life) (Brouard 1986; Guillot 2003). Bongaarts and Feeney (2003) showed equivalence of the adjusted life expectancy at birth, of mean age at death and of CAL under their proportionality assumption. This is clear from (26), as under the proportionality assumption the observed mean age at death coincides with the adjusted mean age at death, which is presented by the first integral in (26). Under the shifting hypothesis, Bongaarts and Feeney (2002) showed that the adjustment coefficient for mortality tempo is determined by change in the adjusted life expectancy. Guillot (2003, p. 53, 2006, p. 16) derived similar relation involving the CAL and without assuming the shifting assumption. Both these results are equivalent to (20), which may be applied to the TMR under assumption (25). Wilmoth (2005) developed general theory based on age-specific adjustments related to the cumulated cohort proportions of deaths (see also Feeney 2006). His results are identical to ours under assumption (25).

6. EMPIRICAL ILLUSTRATION

Our illustration is based on fertility data on first order births in the Czech Republic for 1961-2007[12], which represents a clear case of postponement of births accompanied by rapid changes of the period conditions. Czech fertility after 1993 was considered “a model case of ‘extreme’ situations” with respect to the tempo adjustment (Sobotka 2004, p. 111).

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[12] I thank Tomas Sobotka for providing his estimates for order-specific fertility rates.

Bongaarts-Feeney adjustments of TFR_1 as well as those based on cumulated period proportions of unadjusted rates are presented in Fig. 2. Our theory above implies that both adjustments are only approximations to the true adjustment revealing the period conditions. The first (B-F) adjustment is based on approximating the change of the adjusted mean age at birth in (20) by the observed change of the unadjusted mean age, while the second adjustment is based on approximating the adjusted proportions in (19) by the observed ones. Both adjustments must coincide and provide a correct account for the period conditions if the Bongaarts and Feeney's shifting assumption holds true. If this assumption is significantly violated, however, it is most likely that the two adjustments will differ, as the second one will reflect age irregularities in the tempo change. In this context, it is notable that the two adjustments were nearly identical in the more stable period prior to 1990s, while diverged significantly in the subsequent period of more turbulent changes. (Other empirical exercises also show that these adjustments are close to each other at relatively stable periods, see Ediev 2008.) Hence, *similarity between the two adjustments may be used as an important indication of relevance of the shifting assumption*.

Turning to the case of turbulent changes in tempo and quantum, it is useful to study age patterns of the adjusted rates. Such patterns are presented in Fig. 3: observed in 2007, adjusted according to the Bongaarts-Feeney shifting assumption (i.e. by scaling the observed schedule), and according to the true adjusted rates (23) consistent with the Bongaarts-Feeney adjustment of the TFR1. Adjustments based on (23) show less significant tempo effect at young ages, which is rather natural as teenage fertility may not be reasonably assumed to be postponed to older ages. A better assumption would be to assume that teenage fertility is primarily changing its quantum rather than its tempo in varying environment. At advanced ages, adjustment (23) consistent with the Bongaarts-Feeney estimate also tends to show a less pronounced tempo effect. On the other hand, we find more significant tempo effect at middle childbearing ages, where people do change their childbearing timing more easily.

These considerations provide rational for another adjustment assuming that different age groups respond differently to changing external conditions. (Such possibility is facilitated by the results above on applicability of the period-proportionality assumption to subsets of the life span.) An example of such adjustment based on cumulated period proportions of fertility rates calculated independently below and above age 20 (starting accumulation, correspondingly, from age 12 and from age 20) is presented in Fig. 4. Such a stylized stratification removes the counterintuitive dynamics of the B-F adjustment: our adjusted TFR1 declines deeper in 1990s when period conditions were worst, and rises afterwards.

Figure 2 Observed period (TFR), adjusted according to the cumulated period proportions (TFR*) and Bongaarts-Feeney adjusted (TFR*_BF) total fertility rates for first order births, Czech Republic, 1961-2007.

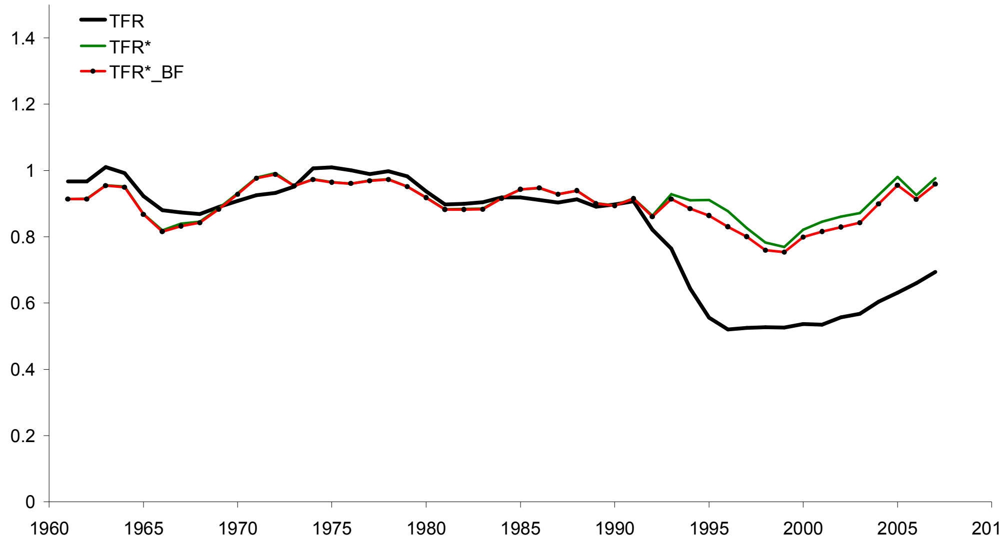


Figure 3 Age schedules of 1st order birth rates, Czech Rep., 2007: observed; proportionately scaled according to the B-F shifting assumption; and the true adjustment (23) consistent with B-F adjustment.

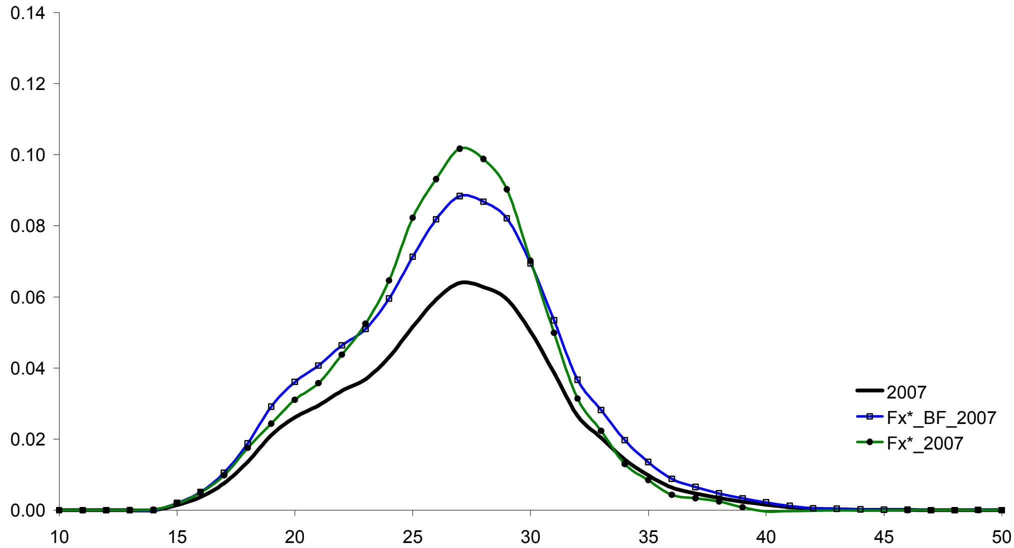
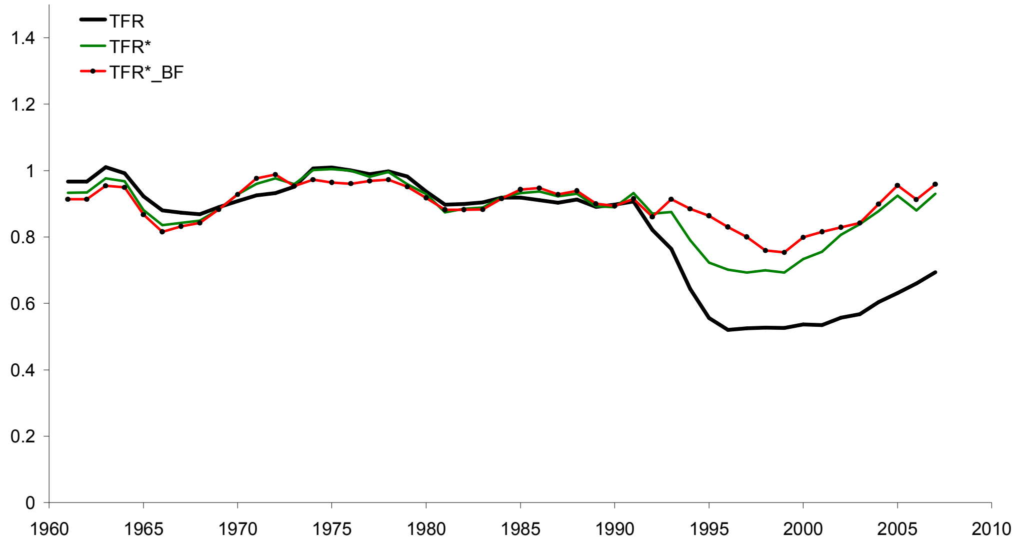


Figure 4 Observed period (TFR), adjusted according to the cumulated period proportions below and above the age 20 (TFR*) and Bongaarts-Feeney adjusted (TFR*_BF) total fertility rates for first order births, Czech Republic, 1961-2007.



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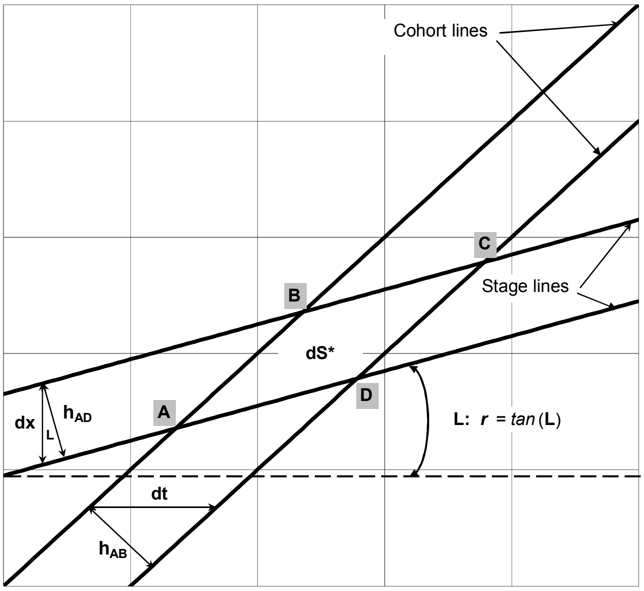
APPENDIX 1: DERIVATION OF THE FORMULAS FOR AREAS BOUNDED BY THE STAGE LINES AND BY PERIOD/COHORT LINES

A simpler case is the case of the area between stage lines and calendar period lines (Fig. 1b). In this case, the area of the parallelogram **EFGH** in Fig. 1b is a product of its base **FE**, i.e. the age increment dx corresponding to the elementary life stage, by the vertical to the base, i.e. the distance dt between time lines. Hence,

$$dS = dxdt\,. \tag{A1.1}$$

(Note, we consider life stages and time/age intervals of infinitesimal duration. Hence, the stage lines are straight and parallel within the elementary periods considered.)

The area between the stage lines and the cohort lines demands more attention (Fig. 1a reproduced below with some additions):



In this case, the quadrangle **ABCD** has no readily known vertical that could be used in calculations. The vertical to the side **AB**, i.e., the distance between the cohort lines, may be obtained from the fact that cohort lines have a slope angle $\pi/4$ to the horizontal, along which the distance between them is given by dt, i.e.

$$h_{AB} = \sin\left(\tfrac{\pi}{4}\right)dt = \frac{dt}{\sqrt{2}}\,. \tag{A1.1}$$

To find the area of the quadrangle, we also need the length of the side **AB** itself. First, we find the distance between the stage lines, i.e. the vertical to the side **AD**. If L is the slope angle of the stage lines, then the same angle is formed by the calendar period lines (along which the distance between the stage lines is dx) and the vertical h_{AD}, see the figure above. Hence,

$$h_{AD} = dx \cdot \cos L\,. \tag{A1.2}$$

One may also note that the angle between sides **AB** and **AD** equals $\dfrac{\pi}{2} - L$ and, therefore,

$$AB = \frac{h_{AD}}{\sin\left(\frac{\pi}{4} - L\right)} = \frac{\cos L}{\sin\left(\frac{\pi}{4} - L\right)} dx \,. \tag{A1.3}$$

Finally, the area may be found as the product of the side and the vertical to it:

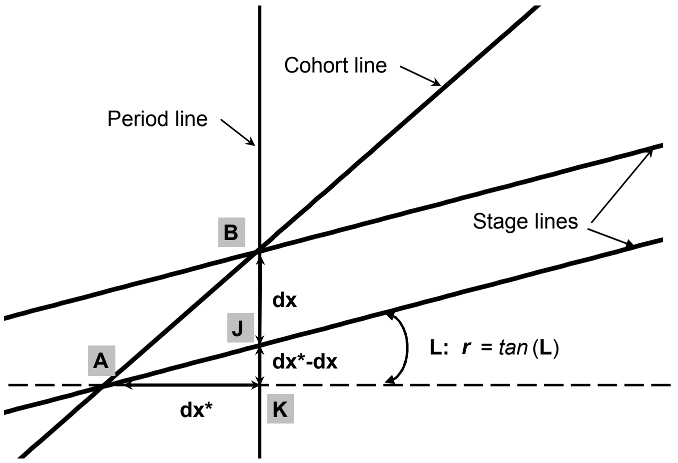
$$dS^* = AB \cdot h_{AB} = \frac{dxdt \cos L}{\sqrt{2}\sin\left(\frac{\pi}{4} - L\right)} = \frac{dxdt \cos L}{\sqrt{2}\left(\sin\frac{\pi}{4}\cos L - \sin L \cos\frac{\pi}{4}\right)} =$$

$$= \frac{dxdt}{1 - \tan L}, \tag{A1.4}$$

and the local adjustment necessary for reconciling the areas from period and cohort perspectives is given by the relation:

$$dS^* = \frac{dS}{1 - \tan L} \,. \tag{A1.5}$$

One may also interpret the relation between cohort- and period-wise exposures to the elementary life stage considering exact birth cohort and exact calendar time point (see fig. below). The exact synthetic cohort observed at calendar time will be exposed to the elementary life stage during the age interval dx (piece **BJ** in the figure), while the exposure of exact birth cohort will be given by age interval dx^* (=**BK**=**AK**). Finally, $JK = dx^* - dx = dx^* \cdot \tan L$ and $dx^* = \frac{dx}{1 - \tan L}$, which is similar to (A1.5).



APPENDIX 2: DERIVATION OF THE ADJUSTMENT FORMULA (3)

Let us denote by $x(s,t)$ the age by which the cohort born at time *t* reaches the stage line assigned to the value s. Similarly, by $x(s;t)$ we denote the age at which the stage line nominated by the value s passes the calendar time *t*. $x(s,t)$ and $x(s;t)$ are two distinct functions linked through our comma-semicolon notations: $x(s,t) = x(s;t + x(s,t))$, $x(s;t) = x(s,t - x(s;t))$.

The slope of the stage line passing at age *x* at calendar time *t*, which appears in the adjustment (1), is given by the partial derivative:

$$r(x;t) \overset{def}{=} r(x,t-x) = \frac{\partial x(s(x;t);t)}{\partial t}. \tag{A2.1}$$

To obtain a more practical formula for this derivative, the following relation may be used (be reminded that expressions with comma and semicolon represent different functions):

$$\begin{aligned} \frac{\partial x(s,t)}{\partial s} &= \frac{dx(s;t+x(s,t))}{ds} = \frac{\partial x(s;t+x(s,t))}{\partial s} + \frac{\partial x(s;t+x(s,t))}{\partial t}\frac{\partial x(s,t)}{\partial s} = \\ &= \frac{\partial x(s;t+x(s,t))}{\partial s} + r(x(s,t);t+x(s,t))\frac{\partial x(s,t)}{\partial s}, \end{aligned} \tag{A2.2}$$

i.e.,

$$\frac{1}{1-r(x(s,t),t)} = \frac{\left(\dfrac{\partial x(s,t)}{\partial s}\right)}{\left(\dfrac{\partial x(s;t+x(s,t))}{\partial s}\right)}. \tag{A2.3}$$

The usual relation for the derivative of inverse function implies that

$$\frac{\partial x(s,t)}{\partial s} = \left[\frac{\partial s(x(s,t),t)}{\partial x}\right]^{-1} \quad \text{and} \quad \frac{\partial x(s;t)}{\partial s} = \left[\frac{\partial s(x(s;t);t)}{\partial x}\right]^{-1}. \tag{A2.4}$$

Combining this with (A2.3) yields:

$$\frac{1}{1-r(x,t)} = \frac{s'_x(x;t)}{s'_x(x,t-x)}, \tag{A2.5}$$

which concludes derivation of the first part of (3). The latter part of Eq. (3) involves the following link between partial derivatives:

$$s'_x(x,t) = \frac{d}{dx}s(x;t+x) = s'_x(x;t+x) + s'_t(x;t+x), \tag{A2.6}$$

which, written for the cohort born at time $t-x$ implies:

$$s'_x(x,t-x) = s'_x(x;t) + s'_t(x;t). \tag{A2.7}$$

APPENDIX 3: DERIVATION OF THE ADJUSTMENT FORMULA IN CASE OF PERIOD-WISE PROPORTIONAL CHANGES OF THE QUANTUM FUNCTION

When cumulated adjusted proportions of life cycle events by age x at calendar time t are used as the stage function, $s(x;t)=p^*(x;t)$, one may derive the following relation for the observed (unadjusted) period total of life cycle events (note, $e^*(x;t)=s'_x(x;t)\cdot TE^*(t)$; $s'_x(x;t)=s'_x(x,t-x)-s'_t(x,t-x)=s'_x(x,t-x)-s'_t(x;t)$):

$$
\begin{aligned}
TE(t) &= \int_0^\infty e(x;t)dx = \int_0^\infty e^*(x;t)\frac{s'_x(x,t-x)}{s'_x(x;t)}dx = \int_0^\infty TE^*(t)s'_x(x,t-x)dx = \\
&= TE^*(t)\int_0^\infty \left(s'_x(x;t)+s'_t(x;t)\right)dx = TE^*(t)\int_0^\infty s'_x(x;t)dx + TE^*(t)\int_0^\infty s'_t(x;t)dx = \\
&= TE^*(t)+TE^*(t)\frac{d}{dt}\int_0^X s(x;t)dx = TE^*(t)-TE^*(t)\frac{d}{dt}\mu^*(t),
\end{aligned} \tag{A2.1}
$$

where X is the upper age limit, after which no life cycle events occur (i.e. $s'_t(x;t)\equiv 0$, $x\geq X$, which allows differentiating by time outside the integral in (A2.1)) and $\mu^*(t)=X-\int_0^X s(x;t)dx=\int_0^X x\cdot s'_x(x;t)dx=\int_0^X x\cdot\frac{e^*(x;t)}{TE^*(t)}dx$ is the mean age at life course event according to the *adjusted period distribution*. Hence,

$$
TE^*(t)=\frac{TE(t)}{1-\frac{d}{dt}\mu^*(t)}. \tag{A2.2}
$$

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